Harmonic Numbers and Their p-Adic Structure

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Contents

p-Adic Valuation

Definition (p-Adic Valuation)

The *p-adic valuation* on $\mathbb{Z}$ is defined by a function $\nu_p : \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$, which $\nu_p(n)$ gives the highest power of a prime p dividing $n \in \mathbb{Z}$. More formally:

$$n = p^{\nu_p(n)}x, \text{ where } p \nmid x. \tag{1}$$

For all nonzero $k \in \mathbb{Q}$, we can write $k = \frac{a}{b}$, where $a, b \in \mathbb{Z}$. Then we can define

$$\nu_p(k) = \nu_p(a) - \nu_p(b). \tag{2}$$

Lastly, we must define $\nu_p(0) = \infty$ (This is the main reason for taking $\mathbb{Z} \cup \{\infty\}$ for the image set of the function ν_p.).

p-Adic Valuation

Remark 1.

By using the second part of the definition, we can easily extend our valuation map $\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$, that is to say, this is merely the generalization of p-adic valuation to rational numbers.

Intuitive Approach on p-Adic Valuation

For a more intuitive approach, we can think of p-adic valuation as how divisible a number is by any prime p. As the assumption of $\nu_p(0) = \infty$ asserts that number 0 is divisible by all powers of a prime p.

Properties of p-Adic Valuation

Lemma 1.

The following properties hold for all $n, m \in \mathbb{Q}$:

1. $\nu_p(nm) = \nu_p(n) + \nu_p(m)$,
2. $\nu_p(n + m) \geq \min\{\nu_p(n), \nu_p(m)\}$,
3. if $\nu_p(n) \neq \nu_p(m)$, then $\nu_p(n + m) = \min\{\nu_p(n), \nu_p(m)\}$.

p-Adic Absolute Value and Metric

Definition (p-Adic Absolute Value)

The *p-adic absolute value* $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ defined as:

$$|n|_p = \begin{cases} p^{-\nu_p(n)} & \text{if } n \neq 0, \\ 0 & \text{if } n = 0. \end{cases},$$

where $n \in \mathbb{Q}$.

Proposition (p-Adic Metric)

The p-adic absolute value $|\cdot|_p$ induces a metric $d_p(x, y) = |x - y|_p$ on $\mathbb{Q}$; moreover, it induces ultrametric on $\mathbb{Q}$. (Ultrametric is a more powerful version of the standard definition of a metric, which changes triangle inequality with Non-Archimedean triangle inequality $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.)

Field of p-Adic Numbers

Remark 2.

$\mathbb{Q}$ is not complete under the p-adic metric d_p. So this raises a natural question: What is the completion?

Definition (Field of p-Adic Numbers - $\mathbb{Q}_p$)

The *field of p-adic numbers* $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic metric d_p.

Weird Fact About p-Adic Completion

Theorem (Ostrowski, 1916)

If $|\cdot|$ is a nontrivial absolute value on $\mathbb{Q}$, then it is either $|\cdot| = |\cdot|_\infty$ or $|\cdot| = |\cdot|_p$ for a prime p (up to equivalence).

Corollary of Ostrowski's Theorem

The only completions of $\mathbb{Q}$ with respect to nontrivial absolute values are the real numbers $\mathbb{R}$ and the p-adic numbers $\mathbb{Q}_p$ for primes p (up to equivalence).

Ring of p-Adic Integers

Definition (Ring of p-Adic Integers - $\mathbb{Z}_p$)

Let the *ring of p-adic integers* $\mathbb{Z}_p$ defined as follows:

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Remark 3.

We note that for all $n \in \mathbb{Z}$, $\nu_p(n) \geq 0$, so this implies that $p^{-\nu_p(n)} = |n|_p \leq 1$. Thus, $\mathbb{Z} \subset \mathbb{Z}_p$.

Idea of Harmonic Numbers

Definition (Harmonic Series)

The *harmonic series* is defined by the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

This series diverges.

Adding the first n terms of this series gives us a partial sum:

Definition (Harmonic Numbers - H_n)

Let n be a positive integer. n^{th} *harmonic number* H_n is defined as

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

Examples of Harmonic Numbers

n	H_n
1	1
2	3/2
3	11/6
4	25/12
5	137/60
6	49/20
7	363/140
8	761/280
9	7129/2520
10	7381/2520
11	83711/27720

Table: First Eleven Harmonic Numbers

Harmonic Numbers

Arithmetic Behaviour of Harmonic Numbers

One can check the analytic properties of this series and harmonic numbers and their connection to γ. In this seminar, we will be looking into arithmetic behaviour, i.e., their p-adic properties as Conrad said in his note "The p-Adic Growth of Harmonic Numbers" [6].

Structure to Analyse Harmonic Numbers

- Nonintegerness of Harmonic Numbers,
- Form and Growth of $J(p)$ Sets,
- Various Theorems About Harmonic Numbers.

Remark 4.

One of the most important properties of harmonic numbers H_n is their nonintegerness when $n > 1$ (It is trivial that $H_1 = \frac{1}{1} = 1$). Theisinger [18] gave a proof of this theorem with a different point of view, but we used the Kürschák's idea [12] of using p-adic valuations.

Nonintegerness of Harmonic Numbers

Theorem (Nonintegerness of H_n)

For $n \geq 2$, $H_n \notin \mathbb{Z}$.

Proof.

Let $2^r \leq n < 2^{r+1}$, so $r \geq 1$ and the highest power of 2 that appears in some reciprocal in the sum defining H_n is 2^r. Only reciprocal in H_n with denominator divisible by 2^r is $\frac{1}{2^r}$. If there exists any different reciprocal, it should be written as $\frac{1}{2^r c}$ for odd $c > 1$, but since if those terms are in the sum, then so is $\frac{1}{2^r 2}$, which is false since $2^{r+1} > 0$. Therefore, $\frac{1}{2^r}$ has a more highly negative 2-adic valuation than every other term in H_n. So it is not cancelled out in the sum. This means that $H_n \notin \mathbb{Z}_2$, so $H_n \notin \mathbb{Z}$ by Remark 3.

Nonintegerness of Harmonic Numbers

Theorem (Nonintegerness of $H_n - H_m$)

For $m \leq n-2, H_n - H_m \notin \mathbb{Z}_2$. In particular, $H_n - H_m \notin \mathbb{Z}$.

Before proceeding to the proof, we can directly see the degenerate cases by taking $m = 1$ and $n \geq 3$, we can recover theorem about nonintegerness of H_n for $n \geq 3$ since $H_1 \in \mathbb{Z}$. This theorem is false if $m = n-1$ and n is odd, since then $H_n - H_m = \frac{1}{n} \in \mathbb{Z}_2$.

Proof.

We can write the difference as:

$$H_n - H_m = \sum_{k=m+1}^{n} \frac{1}{k}.$$

We will show there is a unique term in the sum with the most negative 2-adic valuation, like the proof of nonintegerness of H_n. Let $r = \max\{\nu_2(k)\}$ where $m < k \leq n$. Since $n \geq m+2$, the sum $H_n - H_m$ has at least two terms in it, so some k is even and thus $r \geq 1$.
We will show there is only one integer from $m+1$ up to n with 2-adic valuation r. Suppose there are two such numbers. Write them as $2^r c$ and $2^r d$ with (wlog) odd $c < d$. Then $c+1$ is even and $2^r c < 2^r(c+1) < 2^r d$, so $\frac{1}{(2^r(c+1))}$ appears in $H_n - H_m$. But $\nu_2(2^r(c+1)) \geq r+1$ since c is odd. This contradicts the maximality of r. Therefore, there is only one term in $H_n - H_m$ with 2-adic valuation $-r$, so $\nu_2(H_n - H_m) = -r$.

p-Adic Formulation of Harmonic Numbers

p-Adic Formulation of H_n

Theorem about nonintegerness of harmonic numbers H_n gives us an important formula

$$\nu_2(H_n) = -r, \text{ where } 2^r \leq n < 2^{r+1}.$$

p-Adic Formulation of $H_n - H_m$

Theorem about nonintegerness of $H_n - H_m$ give us a formula

$$\nu_2(H_n - H_m) = -r, \text{ where } \; m \leq n-2 \; \text{ and } \; r = \max_{m<k\leq n} \{\nu_2(k)\},$$

which $k \in H_n - H_m$.

$J(p)$ and $I(p)$ Sets

Eswarathasan and Levine's [9] Important Definitions

To ensure the completeness of the discussion, we must write our H_n as

$$H_n = H(n) = \frac{a(n)}{b(n)},$$

where $a(n), b(n) \in \mathbb{Z}_{\geq 0}, (a(n), b(n)) = 1$. By using these functions, we can define the aforementioned sets

$$J_p = J(p) = \{n \geq 0 : a(n) \equiv 0 \pmod{p}\},$$
$$I_p = I(p) = \{n \geq 0 : b(n) \not\equiv 0 \pmod{p}\}.$$

Conjecture (Eswarathasan and Levine)

For all primes p, $J(p)$ is finite.

Form and A Little of the History of $J(p)$

Even Eswarathasan and Levine (conjecture is given in the literature by them) said that "... it even seems quite difficult to show that $J(11)$ is finite.".

Eswarathasan and Levine (p-Integral Harmonic Sums)

- $J(p)$ sets for primes less than 11
- Recursive form to construct the $J(p)$ sets
- $\{0, p-1, p(p-1), p^2-1\} \subset J(p)$.
- Harmonic Primes and a new conjecture

Form and A Little of the History of $J(p)$

Boyd (A p-Adic Study of the Partial Sums of the Harmonic Series)

- Finiteness of $J(11)$
- Computational Approach
- All $J(p)$ sets for all $p < 550$ with three exceptions: 83, 127, and 397

Proposition (Boyd, 1994)

For any prime $p \geq 3$, the set $J(p)$ is finite if and only if $\nu_p(H_n) \longrightarrow -\infty$ as $n \longrightarrow \infty$.

Remark 5.

With this proposition, he actually gave "p-adic growth of harmonic numbers" expression a meaning. Proof of this can be read from Boyd [5] and Conrad's notes [6].

Various Theorems (Babbage's Theorem)

Firstly, we will go on with Babbage's Theorem [4]:

Theorem (Babbage, 1819)

For each odd prime p, $H_{p-1} \equiv 0 \pmod{p}$.

Remark 6.

With this theorem, we can say that all $J(p)$ sets have at least one element in them, that is to say, for any prime p, $J(p) \neq \emptyset$.

Various Theorems (Wolstenholme's Theorem)

More powerful version of Babbage's Theorem was given in Wolstenholme's paper [19]:

Theorem (Wolstenholme, 1862)

For each prime $p \geq 5, H_{p-1} \equiv 0 \pmod{p^2}$

Proof.

We group the terms in H_{p-1} that are equidistant from the middle of the sum:

$$\begin{aligned} H_{p-1} &= 1 + \frac{1}{2} + \cdots + \frac{1}{p-1} \\ &= \left(1 + \frac{1}{p-1}\right) + \left(2 + \frac{1}{p-2}\right) + \cdots + \left(\frac{1}{(p-1)/2} + \frac{1}{(p-1)/2}\right) \\ &= \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k} + \frac{1}{p-k}\right) = \sum_{k=1}^{(p-1)/2} \frac{p}{k(p-k)} = p \sum_{k=1}^{(p-1)/2} \frac{1}{k(p-k)}. \end{aligned}$$

Various Theorems (Wolstenholme's Theorem)

Proof (Cont.)

Since a p has been pulled out of the sum, to show that $H_{p-1} \equiv 0 \pmod{p^2}$, we will show that $\sum_{k=1}^{(p-1)/2} \frac{1}{k(p-k)} \equiv 0 \pmod{p}$. Like the same argument in Babbage's Theorem, if we reduce the sum $\pmod{p}$, we can get

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k(p-k)} \equiv \sum_{k=1}^{(p-1)/2} \frac{1}{k(-k)} \equiv -\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \quad \text{in } \mathbb{F}_p.$$

Proof (Cont.)

The numbers $1^2, \ldots, ((p-1)/2)^2$ represents the nonzero squares modulo p, so their reciprocals also represent the nonzero squares modulo p. Therefore,

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv \sum_{k=1}^{(p-1)/2} k^2 \quad \text{in } \mathbb{F}_p.$$

Using the well-known formula $S_2(n) = \sum_{k=1}^{n} k^2 = n(n+1)(n+2)/6$ with $n = (p-1)/2$,

$$\sum_{k=1}^{(p-1)/2} k^2 = \frac{p(p^2-1)}{24}.$$

Since $p > 3, (p, 24) = 1$, so this sum is in $\mathbb{F}_p$ and therefore, $H_{p-1} \equiv 0 \pmod{p^2}$.

Various Theorems (Leudesdorf's Theorem)

A generalization of Wolstenholme's Theorem is given in Leudesdorf's paper "Some Results in the Elementary Theory of Numbers" [13]:

Theorem (Leudesdorf, 1889)

Let n be an integer that is coprime with 6,

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n-1} \frac{1}{k} \equiv 0 \pmod{n^2}$$

Definition (Generalized Harmonic Numbers)

The n^{th} *generalized harmonic number of order m* is given by

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}.$$

Remark 7.

The case $m = 1$ reduces to standard harmonic numbers directly. More importantly, we can see generalized harmonic numbers as partial sums of the Riemann zeta function: The limit of $H_{n,m}$ as $n \to \infty$ is finite if $m > 1$, with the generalized harmonic number bounded by and converging to the *Riemann zeta function*

$$\lim_{n \to \infty} H_{n,m} = \zeta(m).$$

Hyperharmonic Numbers

Definition (Hyperharmonic Numbers)

The n^{th} *hyperharmonic number of order* r is defined recursively as

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)},$$

where $H_n^{(0)} = \frac{1}{n}$ and the case $r = 1$ amounts to standard harmonic numbers.

Remark 8.

Some important modern results about the integerness of hyperharmonic numbers are given by Göral, Sertbaş, and Alkan in various papers of theirs [10],[1],[17].

Book Stacking Problem

The *book stacking problem*, also known as the *block stacking problem* or the *leaning tower problem*, asks for the maximum possible overhang that can be achieved by stacking N identical uniform blocks of equal length one on top of another, placed on the edge of a table, without the stack toppling over.

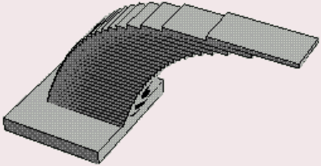

Figure: The Book Stacking Problem by Robert Dickau [8]

Book Stacking Problem

Formula for Overhang

$$\text{Maximum Overhang} = \sum_{i=1}^{N} \frac{1}{2i} = \frac{1}{2} H_N$$

times the width of a single block.

Remark 8.

This expression is exactly one half of the N-th harmonic number. Since the harmonic series diverges, the maximum achievable overhang grows without bound as N increases. In other words, it is theoretically possible to obtain an arbitrarily large overhang by using a sufficiently large number of blocks.

References I

- E. Alkan, H. Göral, and D. C. Sertbaş, "Hyperharmonic Numbers Can Rarely Be Integers," *INTEGERS: Electronic Journal of Combinatorial Number Theory*, vol. 18, paper A43, 2018. DOI: 10.5281/zenodo.10677684.
- Ç. Altuntaş, "On the finiteness of some p-divisible sets," *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 73, pp. 1011–1039, 2024. DOI: 10.31801/cfsuasmas.1441894.
- Ç. Altuntaş and H. Göral, "Dedekind harmonic numbers," *Proceedings of the Indian Academy of Sciences: Mathematical Sciences*, vol. 131, no. 2, Article 46, 2021. DOI: 10.1007/s12044-021-00643-6.
- C. Babbage, "Demonstration of a Theorem Relating to Prime Numbers," *Edinburgh Philosophical Journal*, vol. 1, pp. 46–49, 1819.

References II

D. W. Boyd, "A *p*-adic Study of the Partial Sums of the Harmonic Series," *Experimental Mathematics*, vol. 3, pp. 287–302, 1994. Available at: https://api.semanticscholar.org/CorpusID:17442396.

K. Conrad, "*p*-adic Growth of Harmonic Numbers," 2006. Available on Keith Conrad's expository papers webpage: https://kconrad.math.uconn.edu/blurbs/.

J. H. Conway and R. K. Guy, *The Book of Numbers*. Springer-Verlag, New York, 1996.

R. Dickau, "Harmonic Numbers and the Book-Stacking Problem," 1998. Available at: https://www.robertdickau.com/BookStacking.html. Accessed: 2025-11-10.

References III

- A. Eswarathasan and E. Levine, "p-Integral harmonic sums," *Discrete Mathematics*, vol. 91, no. 3, pp. 249–257, 1991. DOI: 10.1016/0012-365X(90)90234-9.
- H. Göral and D. C. Sertbaş, "Almost all hyperharmonic numbers are not integers," *Journal of Number Theory*, vol. 171, pp. 495–526, 2017. DOI: 10.1016/j.jnt.2016.07.023.
- H. Göral and D. Sertbaş, "Applications of class numbers and Bernoulli numbers to harmonic type sums," *Bulletin of the Korean Mathematical Society*, 2021. DOI: 10.4134/BKMS.b201045.
- J. Kürschák, "On the harmonic series," *Matematikai és Fizikai Lapok*, vol. 27, pp. 299–300, 1918.

References IV

C. Leudesdorf, "Some Results in the Elementary Theory of Numbers," *Proceedings of the London Mathematical Society*, vol. s1-20, no. 1, pp. 199–212, 1888. DOI: 10.1112/plms/s1-20.1.199.

R. Mestrovic, "Wolstenholme's theorem: Its Generalizations and Extensions in the last hundred and fifty years (1862–2012)," arXiv preprint arXiv:1111.3057, 2011.

A. Ostrowski, "Über einige Lösungen der Funktionalgleichung $\varphi(x) \cdot \varphi(y) = \varphi(xy)$," *Acta Mathematica*, vol. 41, pp. 271–284, 1916.

A. Pomerantz, *An Introduction to the p-Adic Numbers*, 2020. Available at: `https://api.semanticscholar.org/CorpusID:225101559`.

References V

- D. C. Sertbaş, "Hyperharmonic integers exist," *Comptes Rendus. Mathématique*, vol. 358, pp. 1179–1185, 2021. DOI: 10.5802/crmath.137.
- L. Theisinger, "Bemerkung über die harmonische Reihe," *Monatshefte für Mathematik und Physik*, vol. 26, pp. 132–134, 1915.
- J. Wolstenholme, "On Certain Properties of Prime Numbers," *The Quarterly Journal of Pure and Applied Mathematics*, vol. 5, pp. 35–39, 1862.

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